Instructions :

1. Answer all questions.
2. Write your answers according to the instructions given below with the questions.
3. Begin each section on a new page.

## SECTION - A

- Given below are 1 to 15 multiple choice questions. Each carries one mark. Write the serial number ( a or b or c or d ) in your answer book of the alternative which you feel is the correct answer of the question.

1. If the co-ordinates of a point ( 6, -1 ) changes to ( 8, - 4 ), then write the co-ordinates of the point where the origin is shifted.

( a ) ( 2, - 3 ) ( b ) ( -3, 2 ) ( c ) ( 3, - 2 ) ( d ) ( - 2, 3 )

**Solution :** ( x, y ) = ( 6, - 1 ) are the coordinates of the point before shifting of the origin. ( x', y' ) = (8, - 4 ) are the coordinates of the point after shifting of the origin. ( h, k ) are the coordinates where the origin is shifted. => h = x - x' = 6 - 8 = - 2 and k = y - y' = - 1 - ( - 4 ) = 3.

2. Write the measure of the angle between the lines x + y = 0 and y = [ $\pi$ ]

( a ) $\frac{\pi}{4}$ ( b ) $\frac{\pi}{3}$ ( c ) $\frac{\pi}{2}$ ( d ) 0

**Solution :** Slope of the line x + y = 0 is m = - 1. y = [ $\pi$ ] is a horizontal line. Angle between the horizontal line and the line of slope m is $\tan^{-1} |m| = \tan^{-1} |-1| = \frac{\pi}{4}$.

3. How many tangents to the circle $x^2 + y^2 = 29$ pass through the point ( 5, 2 ) ?

( a ) 0 ( b ) 1 ( c ) 2 ( d ) none of these

**Solution :** The point ( 5, 2 ) satisfies the equation of the circle and hence lies on the circle. Only one tangent can be drawn at a point on the circle.

4. The standard equation of the parabola having vertex at the origin, passing through ( -1, 1 ) and symmetric about Y - axis is

( a ) $y^2 = -x$ ( b ) $x^2 = y$ ( c ) $y^2 = x$ ( d ) $x^2 = -y$

**Solution :** The standard equation of the parabola having vertex at the origin, and symmetric about Y - axis is $x^2 = ky$. If it passes through the point ( -1, 1 ), $(-1)^2 = k(1)$ => k = 1 => the required equation of the parabola is $x^2 = y$.

5. Write the equation of the auxiliary circle of $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

(a) $x^2 + y^2 = -5$   (b) $x^2 + y^2 = 5$   (c) $x^2 + y^2 = 4$   (d) $x^2 + y^2 = 9$

**Solution :** The equation of the auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$. Here, $a^2 = 4$ => the equation of the auxiliary circle of the given hyperbola is $x^2 + y^2 = 4$.

6. Obtain $| \cos\theta \cos\alpha,\ \cos\theta \sin\alpha,\ \sin\theta |$

(a) -1   (b) 0   (c) 1   (d) none of these

**Solution :** $| \cos\theta \cos\alpha,\ \cos\theta \sin\alpha,\ \sin\theta | = \sqrt{\cos^2\theta \cos^2\alpha + \cos^2\theta \sin^2\alpha + \sin^2\theta}$

$= \sqrt{\cos^2\theta (\cos^2\alpha + \sin^2\alpha) + \sin^2\theta} = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$

7. Find the resultant force of (1, 2, -1) and (1, -2, 1)

(a) (2, 0, 0)   (b) (-1, 4, 2)   (c) (2, 4, 2)   (d) (-2, 0, 0)

**Solution :** The resultant of (x, y, z) = (1, 2, -1) and (x', y', z') = (1, -2, 1) is (x + x', y + y', z + z') = (1 + 1, 2 - 2, -1 + 1) = (2, 0, 0).

8. Find the centre of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$.

(a) (-1, -2, -3)   (b) (3, 2, 1)   (c) (1, 2, 3)   (d) (1, 2, -3)

**Solution :** Comparing the given equation of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$ with the general equation of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0$, the centre of the sphere is (-u, -v, -w) = (1, 2, 3).

9. Find $\lim_{x \to 0} \frac{e^{3x} - 1}{x}$

(a) 3   (b) $\frac{1}{3}$   (c) $\log e^e$   (d) $\log e^3$

**Solution :** $\lim_{x \to 0} \frac{e^{3x} - 1}{x} = 3 \lim_{x \to 0} \frac{e^{3x} - 1}{3x} = 3.$

10. Find $\frac{d}{dx}(\sin^2 x)$

(a) $-\sin 2x$   (b) $\cos^2 x$   (c) $\cos 2x$   (d) $\sin 2x$

**Solution :** $\frac{d}{dx}(\sin^2 x) = 2 \sin x . \frac{d}{dx}(\sin x) = 2 \sin x \cos x = \sin 2x.$

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11. Find c, of Mean - value theorem for $f(x) = \log x,\ x \in [1, e]$

(a) $e - 1$ (b) $1 - e$ (c) $1 - \frac{1}{e}$ (d) $\frac{1}{e - 1}$

**Solution :** $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$. By Mean - value theorem, $f'(c) = \frac{f(e) - f(1)}{e - 1}$

$\Rightarrow \frac{1}{c} = \frac{\log e - \log 1}{e - 1} \Rightarrow c = e - 1.$

12. Evaluate $\int \sin^2(2x + 3)\, dx$

(a) $\frac{x}{2} - \frac{1}{8}\sin(4x + 6) + c$ (b) $\frac{x}{2} + \frac{1}{8}\sin(4x + 6) + c$

(c) $\frac{x}{2} - \frac{1}{4}\sin(2x + 3) + c$ (d) none of these

**Solution :**

$$\int \sin^2(2x + 3)\, dx = \frac{1}{2}\int 2\sin^2(2x + 3)\, dx = \frac{1}{2}\int [1 - \cos(4x + 6)]\, dx = \frac{x}{2} - \frac{1}{8}\sin(4x + 6) + c.$$

13. Obtain $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$

(a) -2 (b) 2 (c) 0 (d) 1

**Solution :** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx = 2\int_{0}^{\frac{\pi}{2}} \cos x\, dx$ ($\because$ cosx is an even function)

$= 2 \sin x\ (x = 0 \text{ to } \pi/2) = 2.$

14. Write the degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + xy = 0$.

(a) 3 (b) 2 (c) 1 (d) none of these

**Solution :** If the differential equation is written in the form of a polynomial in derivatives, then the degree of the highest order derivative is called degree of the differential equation. By this definition of the degree of the differential equation, the degree of the given equation is 1.

15. A particle moves on a line and its distance from a fixed point at time t is x where $x = 4t^2 + 2t$, then at $t = 2$, find the acceleration.

(a) 2 (b) 4 (c) 6 (d) 8

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**Solution :** Velocity, $v = \frac{dx}{dt} = 8t + 2$ and acceleration, $a = \frac{dv}{dt} = 8$.

## SECTION B

- **Answer the following 16 to 30 questions. Each question carries one mark. 15**

16. Find the circumcentre of a triangle having vertices ( 0, 0 ), ( 3, 0 ) and ( 0, 4 ).

**Solution :** **Answer : ( 3/2, 2 ).**

Let A ( 0, 0 ), B ( 3, 0 ) and C ( 0, 4 ) be the given vertices. Here, $AB^2 + AC^2 = BC^2 \Rightarrow \Delta ABC$ is a right triangle. Hence, its circumcentre is the mid - point of the hypotenuse, $\overline{BC}$ which is [ ( 3 + 0 ) / 2, ( 0 + 4 ) / 2 ] = ( 3/2, 2 ).

17. Find the length of the tangent from ( 6, - 5 ) to $x^2 + y^2 = 49$.

**Solution :** **Answer : $2\sqrt{3}$ .**

Length of the tangent from ( a, b ) to the circle $x^2 + y^2 = r^2$ is $\sqrt{a^2 + b^2 - r^2}$

=> length of the tangent from ( 6, - 5 ) to $x^2 + y^2 = 49$ is $\sqrt{(6)^2 + (-5)^2 - 49} = 2\sqrt{3}$ .

**OR**

If $12x + 5y + 16 = 0$ and $12x + 5y - 10 = 0$ are tangents of a circle, find the radius of the circle.

**Solution :** **Answer : radius = 1.**

As the given two lines have the same slope - 12/5, they are parallel. Hence, radius of the circle is half the perpendicular distance between the tangents $= (1/2)\,|16 + 10| / \sqrt{(12)^2 + (5)^2} = 1$.

18. If $x + y + k = 0$ is a line containing the focal chord of the parabola $y^2 = 16x$, then find the value of k.

**Solution :** **Answer : k = - 4.**

The focus of the parabola $y^2 = 16x$ is ( 4, 0 ). As the focal chord passes through the focus, 4 + 0 + k = 0 => k = - 4.

19. Find the measure of eccentric angle of the point $(2\sqrt{2}, 2)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{8} = 1$.

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**Solution :** **Answer :** $\frac{\pi}{4}$.

Comparing the given ellipse $\frac{x^2}{16} + \frac{y^2}{8} = 1$ with the standard equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a^2 = 16$ and $b^2 = 8$. => $a = 4$ and $b = 2\sqrt{2}$.

The parametric point of the ellipse, $(a \cos\theta,\ b \sin\theta) = (4\cos\theta,\ 2\sqrt{2}\sin\theta)$.

Comparing this with the given point, $(2\sqrt{2},\ 2)$, $4\cos\theta = 2\sqrt{2}$ and $2\sqrt{2}\sin\theta = 2$.

=> $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$ => eccentric angle $\theta = \frac{\pi}{4}$.

20. Find the direction cosines of $\bar{i} + \bar{j} + \bar{k}$.

**Solution :** **Answer :** Direction cosines of the given vector are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

$\bar{i} + \bar{j} + \bar{k} = (1, 1, 1)$. Magnitude of $\bar{i} + \bar{j} + \bar{k} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

=> unit vector in the direction of the given vector is $\frac{1}{\sqrt{3}}(1, 1, 1)$

=> direction cosines of the given vector are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

21. If $\bar{a} = (1, -1)$ and $\bar{b} = (1, 0)$, find $\cos \bar{a} \wedge \bar{b}$.

**Solution :** **Answer :** $\frac{1}{\sqrt{2}}$.

$$\cos \bar{a} \wedge \bar{b} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{(1, -1)\cdot(1, 0)}{\sqrt{1^2 + (-1)^2}\ \sqrt{1^2 + 0}} = \frac{1}{\sqrt{2}}.$$

22. Find c, if the lines $\frac{x-1}{c} = \frac{y+2}{-2} = \frac{z-3}{4}$ and $\frac{x-5}{1} = \frac{y-3}{1} = \frac{z+1}{c}$ have the same direction.

**Solution :** **Answer :** - 2.

The directions of the given lines are $(c, -2, 4)$ and $(1, 1, c)$.

If they are the same, then $\frac{c}{1} = \frac{-2}{1} = \frac{4}{c}$ => $c = -2$.

23. Find the x - intercept of $2x - 3y + 6z = 12$.

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**Solution :** **Answer : x - intercept = 6.**

To find the x - intercept, putting y = z = 0 in the given equation, 2x = 12 => x = 6.

24. Obtain $\frac{d}{dx}\left[\frac{3^x}{x^3}\right]$ **OR** Obtain $\frac{d}{dx}\left(5\,\text{cosec}^{-1} x\right)$.

**Solution :** **Answer :** $\frac{3^x\,(x\log 3 - 3)}{x^4}$ **OR** $\frac{-5}{|x|\sqrt{x^2-1}}$

$$\frac{d}{dx}\left[\frac{3^x}{x^3}\right] = \frac{x^3\frac{d}{dx}(3^x) - 3^x\frac{d}{dx}(x^3)}{(x^3)^2} = \frac{x^3 . 3^x\log 3 - 3^x . (3x^2)}{x^6} = \frac{3^x\,(x\log 3 - 3)}{x^4}$$

**OR** $\frac{d}{dx}\left(5\,\text{cosec}^{-1} x\right) = 5\frac{d}{dx}\left(\text{cosec}^{-1} x\right) = \frac{-5}{|x|\sqrt{x^2-1}}$.

25. Obtain $\int e^x\left[\frac{1 + x\log x}{x}\right] dx$ **OR** Obtain $\int \cos(\log x)\, dx$.

**Solution :** **Answer :** $e^x . \log x + c$ **OR** $\frac{x}{2}[\cos(\log x) + \sin(\log x)] + c$.

$$\int e^x\left[\frac{1 + x\log x}{x}\right] dx = \int\left[\frac{1}{x} + \log x\right] e^x\, dx = \int\left[\log x + \frac{d}{dx}(\log x)\right] e^x\, dx = e^x . \log x + c.$$

For $\int \cos(\log x)\, dx$, let $\log x = t \Rightarrow x = e^t$ and $dx = e^t dt$. $\therefore \int \cos(\log x)\, dx = \int e^t \cos t\, dt$

$$= \frac{t}{2}(\cos t + \sin t) + c = \frac{x}{2}[\cos(\log x) + \sin(\log x)] + c.$$

26. Obtain the area of the region bounded by y = x, X - axis, x = 0 and x = 3.

**Solution :** **Answer :** $\frac{9}{2}$.

Required area $= \int_0^3 x\, dx = \left(\frac{x^2}{2}\right)[x = 0 \text{ to } x = 3] = \frac{9}{2}$.

27. Find the value of $\int_{-1}^{1} \sin^3 x \cos^4 x\, dx$.

**Solution :** **Answer : 0.**

If $f(x) = \sin^3 x \cos^4 x$, then $f(-x) = \sin^3(-x)\cos^4(-x) = -\sin^3 x \cos^4 x = -f(x)$.

Thus, $f(x) = \sin^3 x \cos^4 x$ is an odd function and $\int_{-a}^{a} f(x) = 0$, if $f(x)$ is an odd function.

$\therefore \int_{-1}^{1} \sin^3 x \cos^4 x \, dx = 0.$

28. Obtain the differential equation of the family of curves $y = a\sin(x + b)$ ( a and b are arbitrary constants ).

**Solution :** **Answer :** $\frac{d^2y}{dx^2} + y = 0.$

$y = a\sin(x + b) \Rightarrow \frac{dy}{dx} = a\cos(x + b) \Rightarrow \frac{d^2y}{dx^2} = -a\sin(x + b) = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0.$

29. A ball is projected vertically upwards with speed 19.6 m/s. Find its maximum height.

**Solution :** **Answer :** 19.6 m.

Maximum height, $H = u^2 / 2g = (19.6)^2 / 2 \times 9.8 = 19.6$ m.

30. If $x = 2 - 3t + 4t^3$, then find the acceleration after 2 seconds.

**Solution :** **Answer :** 48 $m/s^2$.

$x = 2 - 3t + 4t^3 \Rightarrow$ velocity, $v = dx/dt = -3 + 12t^2 \Rightarrow$ acceleration, $a = dv/dt = 24t$
At $t = 2$ sec., acceleration, $a = 24 \times 2 = 48$ $m/s^2$.

## SECTION C

- **Answer the following 31 to 40 questions as directed. Each question carries two marks.** **20**

31. Find the equations of the lines through ( 2, 3 ) and making an angle of measure $\frac{2\pi}{3}$ with the Y - axis.

**Solution :** **Answer :** $\sqrt{3}\,y - 3\sqrt{3} = \pm(x - 2).$

If the lines make an angle of measure $\frac{2\pi}{3}$ with the Y - axis, they make angles of measure $\pm\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \pm\frac{\pi}{6}$. The slopes of these lines are $\pm\tan\left(\frac{\pi}{6}\right) = \pm\frac{1}{\sqrt{3}}$. The equations of lines passing through ( 2, 3 ) having these slopes are $y - 3 = \pm\frac{1}{\sqrt{3}}(x - 2)$.

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32. A quadrilateral ABCD is inscribed in a parabola. The sides $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{DA}$, make angles of measure $\theta_1, \theta_2, \theta_3$ and $\theta_4$ with the axis of the parabola respectively. Prove that $\cot\theta_1 + \cot\theta_3 = \cot\theta_2 + \cot\theta_4$.

**Solution :**
A $(at_1^2, 2at_1)$, B $(at_2^2, 2at_2)$, C $(at_3^2, 2at_3)$, and D $(at_4^2, 2at_4)$ are any four parametric points on parabola $y^2 = 4ax$. Slope of $\overleftrightarrow{AB} = \tan\theta_1 = \dfrac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \dfrac{2}{t_1 + t_2}$ => $\cot\theta_1 = \dfrac{t_1 + t_2}{2}$.
Similarly, it can shown that $\cot\theta_2 = \dfrac{t_2 + t_3}{2}$, $\cot\theta_3 = \dfrac{t_3 + t_4}{2}$ and $\cot\theta_4 = \dfrac{t_4 + t_1}{2}$. => $\cot\theta_1 + \cot\theta_3 = \dfrac{t_1 + t_2}{2} + \dfrac{t_3 + t_4}{2} = \dfrac{t_2 + t_3}{2} + \dfrac{t_4 + t_1}{2} = \cot\theta_2 + \cot\theta_4$. - **Proved.**

33. If the length of the latus - rectum is 4 and distance between the foci is $4\sqrt{2}$, then find the standard equation of the ellipse.

**Solution :** **Answer : $x^2 + 2y^2 = 16$**

For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, length of latus - rectum $= \dfrac{2b^2}{a}$, distance between foci = 2ae and $b^2 = a^2(1 - e^2)$ if $a > b$.
=> $\dfrac{2b^2}{a} = 4$, i.e., $b^2 = 2a$ and $2ae = 4\sqrt{2}$, i.e., $4a^2e^2 = 32$ or $a^2e^2 = 8$
Putting these values of $b^2$ and $a^2e^2$ in $b^2 = a^2(1 - e^2)$ => $2a = a^2 - 8$
=> $a^2 - 2a - 8 = 0$ => $(a - 4)(a + 2) = 0$ => $a = 4$ ( $\because$ a cannot be negative )
$\therefore$ $a^2 = 16$ and $b^2 = 2a = 8$ => equation of the ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{8} = 1$ or $x^2 + 2y^2 = 16$.

**OR**

If the difference between the measures of the eccentric angles of points P and Q of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{\pi}{2}$ and if $\overleftrightarrow{PQ}$ cuts intercepts c and d on the axes, then prove that $\dfrac{a^2}{c^2} + \dfrac{b^2}{d^2} = 2$.

**Solution :** Let $P = (a\cos\theta, b\sin\theta)$. Then Q can be taken $[a\cos(\theta + \pi/2), b\sin(\theta + \pi/2)]$, i.e., $Q = (-a\sin\theta, b\cos\theta)$. As the line $\overleftrightarrow{PQ}$ cuts intercepts c and d on the axes, its equation is $\dfrac{x}{c} + \dfrac{y}{d} = 1$. Putting the co - ordinates of P and Q in this equation of the line, we have $\dfrac{a\cos\theta}{c} + \dfrac{b\sin\theta}{d} = 1$ ... (1) and $-\dfrac{a\sin\theta}{c} + \dfrac{b\cos\theta}{d} = 1$ ... (2). Squarring and adding equations (1) and (2), we get $\dfrac{a^2}{c^2} + \dfrac{b^2}{d^2} = 2$. - **Proved.**

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34. If a line $lx + my + n = 0$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then show that $a^2l^2 - b^2m^2 = n^2$.

**Solution :**
The equation of tangent at the parametric point $(a\sec\theta,\ b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$, i.e., $(b\sec\theta)x - (a\tan\theta)y - ab = 0$.
Comparing this with the given equation, $lx + my + n = 0$,
$\frac{b\sec\theta}{l} = \frac{-a\tan\theta}{m} = \frac{-ab}{n} \Rightarrow \sec\theta = -\frac{al}{n}$ and $\tan\theta = \frac{bm}{n}$. Using the identity,
$\sec^2\theta - \tan^2\theta = 1$, we get , i.e., $a^2l^2 - b^2m^2 = n^2$. **- Proved.**

**OR**

If S and S' are the foci of a rectangular hyperbola with its centre C ( 0, 0 ), then prove that for any point P on the rectangular hyperbola
SP . S'P = $CP^2$.

**Solution :**
For the rectangular hyperbola, $x^2 - y^2 = a^2$, foci are $S(\sqrt{2}a,\ 0)$ and $S'(-\sqrt{2}a,\ 0)$ and any point $P = (a\sec\theta,\ a\tan\theta)$. Let M = foot of perpendicular from P on the directrix of the rectangular hyperbola, $x = -a/\sqrt{2}$. $\Rightarrow M = (-a/\sqrt{2},\ a\tan\theta)$
$SP^2 = e^2 . PM^2 = 2(a\sec\theta - a/\sqrt{2})^2$ and $S'P^2 = 2(a\sec\theta + a/\sqrt{2})^2$
$\Rightarrow SP^2 . S'P^2 = 4(a^2\sec^2\theta - a^2/2)^2$
$\Rightarrow SP . SP' = 2a^2\sec^2\theta - a^2 = a^2\sec^2\theta + a^2\tan^2\theta = CP^2$. **- Proved.**

35. Prove that : $[\bar{x} + \bar{y}\quad \bar{y} + \bar{z}\quad \bar{z} + \bar{x}] = 2[\bar{x}\ \ \bar{y}\ \ \bar{z}]$.

**Solution :** LHS $= [\bar{x} + \bar{y}\quad \bar{y} + \bar{z}\quad \bar{z} + \bar{x}] = (\bar{x} + \bar{y}).[(\bar{y} + \bar{z}) \times (\bar{z} + \bar{x})]$
$= (\bar{x} + \bar{y}).[(\bar{y} \times \bar{z}) + (\bar{y} \times \bar{x}) + (\bar{z} \times \bar{z}) + (\bar{z} \times \bar{x})]$
$= (\bar{x} + \bar{y}).[(\bar{y} \times \bar{z}) + (\bar{y} \times \bar{x}) + (\bar{z} \times \bar{x})]$  $[\because \bar{z} \times \bar{z} = \bar{0}]$
$= \bar{x}.(\bar{y}\times\bar{z}) + \bar{y}.(\bar{y}\times\bar{z}) + \bar{x}.(\bar{y}\times\bar{x}) + \bar{y}.(\bar{y}\times\bar{x}) + \bar{x}.(\bar{z}\times\bar{x}) + \bar{y}.(\bar{z}\times\bar{x})$
$= \bar{x}.(\bar{y}\times\bar{z}) + 0 + 0 + 0 + 0 + \bar{y}.(\bar{z}\times\bar{x}) = 2[\bar{x}\ \ \bar{y}\ \ \bar{z}]$ = RHS **- Proved.**

36. Find a unit vector in $R^3$ making an angle of measure $\frac{\pi}{3}$ with each of the vectors ( 1, - 1, 0 ) and ( 0, 1, 1 ).

**Solution :** **Answer :** $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$.

Let $\bar{a} = (x,\ y,\ z)$ be a unit vector in $R^3$ making an angle of measure $\frac{\pi}{3}$ with each of the vectors, $\bar{b} = (1,\ -1,\ 0)$ and $\bar{c} = (0,\ 1,\ 1)$.

$$\Rightarrow \frac{\bar{a}.\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{\bar{a}.\bar{c}}{|\bar{a}|\,|\bar{c}|} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } |\bar{a}| = 1$$

$$\Rightarrow \frac{(x, y, z).(1, -1, 0)}{1.\sqrt{1^2 + (-1)^2}} = \frac{(x, y, z).(0, 1, 1)}{1.\sqrt{1^2 + (1)^2}} = \frac{1}{2} \Rightarrow \frac{x - y}{\sqrt{2}} = \frac{y + z}{\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow x = y + \frac{1}{\sqrt{2}},\ z = \frac{1}{\sqrt{2}} - y \text{ and } x^2 + y^2 + z^2 = 1$$

$$\Rightarrow \left(y + \frac{1}{\sqrt{2}}\right)^2 + y^2 + \left(\frac{1}{\sqrt{2}} - y\right)^2 = 1 \Rightarrow 3y^2 = 0 \Rightarrow y = 0,\ x = \frac{1}{\sqrt{2}} \text{ and } z = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \text{required unit vector is } \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

37. Find the centre and radius of the sphere $|\bar{r}|^2 - \bar{r}.(6, 12, 14) + 30 = 0$.
Also express this equation in the Cartesian form.

**Solution :** Answer : Centre ( 3, 6, 7 ), radius = 8 and Cartesian equation of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 14z + 30 = 0$.

Putting $\bar{r} = (x, y, z)$, the Cartesian equation of the sphere is $x^2 + y^2 + z^2 - 6x - 12y - 14z + 30 = 0$.
Comparing this with the Cartesian equation of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0$,
Centre of the sphere = ( - u, - v, - w ) = ( 3, 6, 7 )
and radius = $\sqrt{u^2 + v^2 + w^2 - c} = \sqrt{3^2 + 6^2 + 7^2 - 30} = 8$.

38. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, then find $y_2$.

**Solution :** Answer : $\frac{\sec^3\theta}{a\theta}$.

$x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta \text{ and } \frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx} = \frac{d}{d\theta}(\tan\theta).\frac{1}{a\theta\cos\theta} = \sec^2\theta.\frac{1}{a\theta\cos\theta} = \frac{\sec^3\theta}{a\theta}.$$

39. Find c, applying Mean - value theorem to $f(x) = \cos^{-1}x,\ x \in [-1, 0]$

**Solution :** Answer : $-\sqrt{1 - \frac{4}{\pi^2}}$ OR 32, 32.

For $f(x) = \cos^{-1}x,\ f'(x) = -\frac{1}{\sqrt{1 - x^2}}$ According to MV Theorem, $f'(c) = \frac{f(-1) - f(0)}{-1 - 0}$

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$\Rightarrow -\dfrac{1}{\sqrt{1-c^2}} = \dfrac{\cos^{-1}(-1) - \cos^{-1}(0)}{-1} \Rightarrow \dfrac{1}{\sqrt{1-c^2}} = \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2} \Rightarrow 1 - c^2 = \dfrac{2}{\pi}$

$\Rightarrow c = -\sqrt{1-\dfrac{4}{\pi^2}}$ ( because $x \in [-1, 0]$ ).

**OR**

**Divide 64 into two parts such that the sum of their cubes is minimum.**

**Solution :**

Dividing 64 into two parts, x and 64 - x, we have to minimize $x^3 + (64 - x)^3$
Let $f(x) = x^3 + (64 - x)^3 = (3 \times 64)x^2 - (3 \times 64^2)x + (64)^3$
For $f(x)$ to be minimum, $f'(x) = 0$ and $f''(x) > 0$
$f'(x) = 0 \Rightarrow (6 \times 64)x - (3 \times 64^2) = 0 \Rightarrow x = 32$. Also $f''(x) = 6 \times 64 > 0$.
=> 64 should be divided into 32 and 32 so that the sum of their cubes is minimum.

**40.** Obtain $\displaystyle\int \frac{x\,dx}{(1+x^2)(x^2-2)}$ **OR** Obtain $\displaystyle\int \frac{\log x - 1}{(\log x)^2}dx$.

**Solution:** **Answer :** $\left(\dfrac{1}{6}\right)\log\left|\dfrac{x^2-2}{x^2+1}\right| + c$ **OR** $\dfrac{x}{\log x} + c$.

For $\displaystyle\int \frac{x\,dx}{(1+x^2)(x^2-2)}$, let $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow \displaystyle\int \frac{x\,dx}{(1+x^2)(x^2-2)} = \left(\frac{1}{2}\right)\int \frac{dt}{(t+1)(t-2)}$

Let $\dfrac{1}{(t+1)(t-2)} = \dfrac{A}{t+1} + \dfrac{B}{t-2} \Rightarrow 1 = A(t-2) + B(t+1)$

$t = -1 \Rightarrow A = -\dfrac{1}{3}$ and $t = 2 \Rightarrow B = \dfrac{1}{3}$

$\therefore \displaystyle\int \frac{x\,dx}{(1+x^2)(x^2-2)} = \left(\frac{1}{2}\right)\int \frac{dt}{(t+1)(t-2)} = \left(\frac{1}{6}\right)\left[\int \frac{dt}{t-2} - \int \frac{dt}{t+1}\right]$

$= \left(\dfrac{1}{6}\right)\log\left|\dfrac{t-2}{t+1}\right| + c = \left(\dfrac{1}{6}\right)\log\left|\dfrac{x^2-2}{x^2+1}\right| + c$

**OR**

For $\displaystyle\int \frac{\log x - 1}{(\log x)^2}dx$, let $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$\Rightarrow \displaystyle\int \frac{\log x - 1}{(\log x)^2}dx = \int \left(\frac{t-1}{t^2}\right)e^t\,dt = \int \left[\frac{1}{t} + \frac{d}{dt}\left(\frac{1}{t}\right)\right]e^t\,dt = \frac{e^t}{t} + c = \frac{x}{\log x} + c.$

## SECTION D

- **Answer the following 41 to 50 questions as directed. Each question carries 3 marks.** **30**

**41.** If A is ( 1, - 2 ) and B is ( - 7, 1 ), find a point P on $\overleftrightarrow{AB}$ such that 3 AP = 5 AB.

**Solution :** **Answer :** $\left(\frac{43}{3}, -7\right)$, $\left(-\frac{37}{3}, 3\right)$.

3 AP = 5 AB => $\frac{PA}{AB} = \frac{5}{3}$ => A divides $\overline{PB}$ in the ratio 5 : 3 or 5 : - 3 from P.

Let P = ( a, b )

=> for external division 5 : 3, $1 = \frac{3a + 5(-7)}{3 + 5}$ => a = $\frac{43}{3}$ and $-2 = \frac{3b + 5(1)}{3 + 5}$ => b = - 7.

and for internal division, 5 : - 3, $1 = \frac{-3a + 5(-7)}{5 - 3}$ => a = $-\frac{37}{3}$ and $-2 = \frac{-3b + 5(1)}{5 - 3}$ = 3.

**42.** Show that the circles $x^2 + y^2 + 6x + 2y - 90 = 0$ and $x^2 + y^2 - 34x - 28y + 260 = 0$ touch each other externally. Also find the equation of a line that contains the diameters of both the circles.

**Solution :** **Answer :** 3x - 4y + 5 = 0

For the circle, $x^2 + y^2 + 6x + 2y - 90 = 0$, centre is C = ( - 3, - 1 ) and radius $r = \sqrt{3^2 + 1^2 + 90} = 10$

For circle, $x^2 + y^2 - 34x - 28y + 260 = 0$, centre is C' = ( 17, 14 ) and radius $r' = \sqrt{17^2 + 14^2 - 260} = 15$

If the circles touch each other externally, r + r' = CC'. Here, $CC' = \sqrt{(17+3)^2 + (14+1)^2} = 25$ and r + r' = 10 + 15 = 25. This proves that the given circles touch each other externally.

$\overleftrightarrow{CC'}$ is the line containing diameter of both the circles. Slope of $\overleftrightarrow{CC'}$ = (14 + 1) / (17 + 3) = 3/4

As it passes through (- 3, - 1 ), its equation is y + 1 = ( 3/4 ) (x + 3), i.e., 3x - 4y + 5 = 0.

**OR**

If the circle $x^2 + y^2 + 2x + fy + k = 0$ touches both the axes, find f and k.

**Solution :** **Answer :** f = $\pm 2$, k = 1.

Equation of a circle touching both the axes will have centre ( r, r ) and radius r. Hence its equation is $( x \pm r )^2 + ( y \pm r )^2 = r^2$, i.e., $x^2 + y^2 \pm 2rx \pm 2ry + r = 0$. Comparing this with the given equation, $x^2 + y^2 + 2x + fy + k = 0$, $2 = \pm 2r$, $f = \pm 2r$ and $k = r^2$ => $f = \pm 2$, k = 1.

**43.** Use vectors to prove that for a $\Delta$ ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

**Solution :** Suppose $\overrightarrow{AB} = \bar{c}$, $\overrightarrow{BC} = \bar{a}$ and $\overrightarrow{CA} = \bar{b}$ $\therefore$ $\bar{a} + \bar{b} + \bar{c} = \bar{0}$

$\therefore$ $(\bar{a} \times \bar{a}) + (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c}) = \bar{0}$ But $\bar{a} \times \bar{a} = \bar{0}$ and $\bar{a} \times \bar{c} = -\bar{c} \times \bar{a}$

$\therefore$ $|\bar{a} \times \bar{b}| = |\bar{c} \times \bar{a}|$ $\therefore$ ab sin ( $\pi$ - C ) = ac sin ( $\pi$ - B )

$\therefore$ $\frac{b}{\sin B} = \frac{c}{\sin C}$. Similarly, $\frac{a}{\sin A} = \frac{b}{\sin B}$ $\therefore$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

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**OR**

**Prove that the angle in a semicircle is a right angle by vector method.**

**Solution :** Let $\overline{AB}$ be diameter of a circle with centre O. Let A = ( a, 0 ) and B = ( - a, 0 ). Let C = ( b, c ). Radius of the circle, OA = OC => $OA^2 = OC^2$ => $a^2 = b^2 + c^2$.

Now, $\overrightarrow{AC}$ = ( b, c ) - ( a, 0 ) = ( b - a, c ) and $\overrightarrow{BC}$ = ( b, c ) - ( - a, 0 ) = ( b + a, c )

=> $\overrightarrow{AC}.\overrightarrow{BC} = (b - a, c).(b + a, c) = b^2 + c^2 - a^2 = 0$ => $\overline{AC} \perp \overline{BC}$ => $m\angle ACB = \frac{\pi}{2}$. -Proved.

**44.** Find the measure of the angle between two lines, if their direction cosines ( l, m, n ) satisfy $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$.

**Solution :** **Answer :** $\frac{\pi}{3}$.

$l + m + n = 0$ ... ( 1 ) and $l^2 + m^2 - n^2 = 0$ ... ( 2 ). Putting $n = -(l + m)$ from eqn. ( 1 ) in eqn. ( 2 ), $l^2 + m^2 - [-(l + m)]^2 = 0$ => $lm = 0$ => $l = 0$ or $m = 0$.

( i ) If $l = 0$, then from eqn. ( 1 ), $m = -n$ => direction cosines of the first line = ( 0, - n, n ).

( ii ) If $m = 0$, then from eqn. ( 1 ), $l = -n$ => direction cosines of the 2nd line = ( l, 0, - l ).

=> angle between the two lines = $\cos^{-1}\frac{(0, -n, n).(l, 0, -l)}{\sqrt{2n^2}\sqrt{2l^2}} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$.

**45.** Find the image of A ( 2, 3, 2 ) in the plane $\bar{r}$ . ( 1, - 2, 1 ) = - 5.

**Solution:** **Answer :** ( 1, 5, 1 )

Let M be the foot of perpendicular from A to the plane.

Normal $\overrightarrow{AM}$ has the direction ( 1, - 2, 1 ). => equation of $\overleftrightarrow{AM}$ is ( 2, 3, 2 ) + k ( 1, - 2, 1 ), $k \in R$.
For some k, M must be ( k + 2, - 2k + 3, k + 2 ).

As M is on the plane, ( k + 2 ) - 2 ( - 2k + 3 ) + ( k + 2 ) = - 5 => $k = -\frac{1}{2}$ => $M = \left(\frac{3}{2}, 4, \frac{3}{2}\right)$.

If the required image of A is ( x', y' z' ), then M being the mid – point of $\overline{AB}$,

$M = \left(\frac{2 + x'}{2}, \frac{3 + y'}{2}, \frac{2 + z'}{2}\right)$ => $\frac{2 + x'}{2} = \frac{3}{2}$, $\frac{3 + y'}{2} = 4$ and $\frac{2 + z'}{2} = \frac{3}{2}$ => x' = 1, y' = 5, z' = 1

=> the required image of A is B = ( 1, 5, 1 )

**OR**

Find the foot of the perpendicular from ( 2, - 1, 2 ) to 2x - 3y + 4z = 44, the equation of this perpendicular and the perpendicular distance between the point and the plane.

**SOLUTION :** **Answer :** ( 4, - 4, 6 ), $\bar{r} = (2, -1, 2) + k(2, -3, 4)$, $k \in R$.

the perpendicular distance = $\sqrt{29}$.

Direction of normal to the plane is ( 2, - 3, 4 ).

$\therefore$ equation of perpendicular from A ( 2, - 1, 2 ) to the plane is $\bar{r} = (2, -1, 2) + k(2, -3, 4)$, $k \in R$.
$\therefore$ the foot of the perpendicular must be ( 2 + 2k, - 1 - 3k, 2 + 4k ) for some $k \in R$.
As this point is on the given plane, 2 ( 2 + 2k ) + 3 ( 1 + 3k ) + 4 ( 2 + 4k ) = 44 => k = 1.
$\therefore$ the foot of the perpendicular is M ( 4, - 4, 6 ). As A is ( 2, - 1, 2 ), the perpendicular distance, AM = $\sqrt{4 + 9 + 16} = \sqrt{29}$.

46. Obtain $\lim\limits_{x \to 0} \dfrac{(1+mx)^n - (1+nx)^m}{x^2}$, $m, n \in N$.

**Solution :** **Answer :** $\dfrac{mn(n-m)}{2}$.

$(1 + mx)^n = 1 + {}^nC_1 (mx) + {}^nC_2 (mx)^2 + {}^nC_3 (mx)^3 + \dots + {}^nC_n (mx)^n$
$(1 + nx)^m = 1 + {}^mC_1 (nx) + {}^mC_2 (nx)^2 + {}^mC_3 (nx)^3 + \dots + {}^mC_m (nx)^m$
=> $(1 + mx)^n - (1 + nx)^m$
$= [\, 1 + {}^nC_1 (mx) + {}^nC_2 (mx)^2 + {}^nC_3 (mx)^3 + \dots + {}^nC_n (mx)^n \,] - [\, 1 + {}^mC_1 (nx) + {}^mC_2 (nx)^2 + {}^mC_3 (nx)^3 + \dots + {}^mC_m (nx)^m \,]$
$= x^2 \, [\, ( {}^nC_2 m^2 + {}^nC_3 m^3 x + \dots + {}^nC_n m^n x^{n-2} ) - ( {}^mC_2 n^2 + {}^mC_3 n^3 x + \dots + {}^mC_m n^m x^{m-2} ) \,]$

$$\Rightarrow \frac{(1+mx)^n - (1+nx)^m}{x^2} = [\, ( {}^nC_2 m^2 + {}^nC_3 m^3 x + \dots + {}^nC_n m^n x^{n-2} ) - ( {}^mC_2 n^2 + {}^mC_3 n^3 x + \dots + {}^mC_m n^m x^{m-2} ) \,]$$

$$\Rightarrow \lim_{x \to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2},\ m, n \in N = {}^nC_2 m^2 - {}^mC_2 n^2 = \frac{mn(n-1) - mn(m-1)}{2} = \frac{mn(n-m)}{2}.$$

47. Find c, if Rolle's theorem is applicable for $f(x) = \sin^4 x + \cos^4 x$, $x \in \left[0, \dfrac{\pi}{2}\right]$.

**Solution :** **Answer :** $\dfrac{\pi}{4}$.

Here, f is continuous on $\left[0, \dfrac{\pi}{2}\right]$ and differentiable on $\left(0, \dfrac{\pi}{2}\right)$ and $f(0) = f\left(\dfrac{\pi}{2}\right) = 1$.
Now, $f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x = 4 \sin x \cos x (\sin^2 x - \cos^2 x) = 0 \Rightarrow \cos 2x = 0$
$\Rightarrow 2x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{4} \quad \therefore c = \dfrac{\pi}{4} \in \left(0, \dfrac{\pi}{2}\right)$.

48. Obtain $\displaystyle\int_0^1 \frac{\log(1+t)}{(1+t)^2}\, dt$.

**Solution :** **Answer :** $\dfrac{1}{2}(1 + \log 2)$.

$$\int_0^1 \frac{\log(1+t)}{(1+t)^2}\, dt = -\int_0^1 \frac{\log \frac{1}{(1+t)}}{(1+t)^2}\, dt.\ \text{Let } \frac{1}{1+t} = y \quad \therefore -\frac{dt}{(1+t)^2} = dy,\ y = 1 \text{ to } \frac{1}{2}$$

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$$\Rightarrow -\int \frac{\log \frac{1}{1+t}}{(1+t)^2}\,dt = \int \log y\,dy = \log y \int dy - \int\left[\frac{d}{dy}(\log y)\int dy\right]dy = y\log y - y$$

Putting the limits of y = 1 to $\frac{1}{2}$, $\int_0^1 \frac{\log(1+t)}{(1+t)^2}\,dt = \frac{1}{2}(1 + \log 2)$.

49. If the line y = c, divides the area of the region bounded by the parabola $x^2 = 4y$ and the line y = 16 in two equal areas, find the value of c.

**Solution :** **Answer :** $2^{\frac{10}{3}}$

$$\int_0^c x\,dy = \int_c^{16} x\,dy \Rightarrow 2\int_0^c \sqrt{y}\,dy = 2\int_c^{16} \sqrt{y}\,dy \Rightarrow [y^{\frac{3}{2}}]_0^c = [y^{\frac{3}{2}}]_c^{16}$$

$$\Rightarrow c^{\frac{3}{2}} = (16)^{\frac{3}{2}} - c^{\frac{3}{2}} \Rightarrow 2c^{\frac{3}{2}} = (16)^{\frac{3}{2}} \Rightarrow 2c^{\frac{3}{2}} = 64 \Rightarrow c = 2^{\frac{10}{3}}$$

50. Solve : $\frac{dy}{dx} = \sin(x + y)$.

**Solution:** **Answer :** $\tan(x+y) + \sec(x+y) = x + c.$

Let x + y = v => $1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dv}{dx} - 1 = \sin v \Rightarrow \frac{dv}{1+\sin v} = dx$

$$\Rightarrow \int \frac{dv}{1+\sin v} = \int dx \Rightarrow \int \frac{(1-\sin v)\,dv}{1-\sin^2 v} = x + c \Rightarrow \int \frac{(1-\sin v)\,dv}{\cos^2 v} = x + c$$

$$\Rightarrow \int(\sec^2 v + \sec v \tan v)\,dv = x + c \Rightarrow \tan v + \sec v = x + c \Rightarrow \tan(x+y) + \sec(x+y) = x + c.$$

## SECTION E

- **Answer the following 51 to 54 questions. Each question carries 5 marks. 20**

51. Find the equation of the line passing through the origin and containing a line - segment of length $\sqrt{10}$ between the lines 2x - y + 1 = 0 and 2x - y + 6 = 0.

**Solution :** **Answer :** 3x + y = 0 or x - 3y = 0.

Let y = mx be the line passing through the origin. Solving it with the line 2x - y + 1 = 0, 2x - mx + 1 = 0 => x = 1 / (m - 2) and y = m / (m - 2).
Similarly, solving it with the line 2x - y + 6 = 0, x = 6 / (m - 2) and y = 6m / (m - 2).
Thus, [1 / (m - 2), m / (m - 2) ] and [ 6 / (m - 2), 6m / (m - 2) ] are the intersection points.
=> $[6/(m-2) - 1/(m-2)]^2 + [6m/(m-2) - m/(m-2)]^2 = 10$
=> $(25 + 25m^2) = 10(m-2)^2$ => $15m^2 + 40m - 15 = 0$ => $3m^2 + 8m - 3 = 0$
=> (m + 3) (3m - 1) = 0 => m = - 3 or 1/3 => equations of required lines are
y = - 3x or y = (1/3)x, i.e., 3x + y = 0 or x - 3y = 0.

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**OR**

**A ( 1, - 2 ) is a vertex of $\Delta$ ABC. Equations of the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ are x - y + 5 = 0 and x + 2y = 0 respectively. Find the co-ordinates of B and C.**

**<u>Solution :</u>** **Answer : B = ( - 7, 6 ) and C = $\left(\frac{11}{5}, \frac{2}{5}\right)$.**

Let B = ( b, b' ) and C = ( c, c' ).

Mid - point of $\overline{AB}$, M = $\left(\frac{b+1}{2}, \frac{b'-2}{2}\right)$ lies on its bisector, x - y + 5 = 0 => $\frac{b+1}{2} - \frac{b'-2}{2} + 5 = 0$ => b - b' + 13 = 0 … ( 1 ).

( Slope of $\overline{AB}$ ) x ( slope of its perpendicular bisector ) = - 1 => $\left(\frac{b'+2}{b-1}\right) \times 1 = -1$

=> b + b' + 1 = 0 … ( 2 ). Solving equations ( 1 ) and ( 2 ) gives b = - 7 and b' = 6.

Mid - point of $\overline{AC}$, N = $\left(\frac{c+1}{2}, \frac{c'-2}{2}\right)$ lies on its bisector, x + 2y = 0 => $\frac{c+1}{2} + c' - 2 = 0$

=> c + 2c' - 3 = 0 … ( 3 ).

( Slope of $\overline{AC}$ ) x ( slope of its perpendicular bisector ) = - 1 => $\left(\frac{c'+2}{c-1}\right) \times \left(-\frac{1}{2}\right) = -1$

=> 2c - c' - 4 = 0 … ( 4 ). Solving equations ( 3 ) and ( 4 ) gives c = $\frac{11}{5}$ and c' = $\frac{2}{5}$.

=> B = ( - 7, 6 ) and C = $\left(\frac{11}{5}, \frac{2}{5}\right)$.

**52. Prove that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a bounded sequence ( where n $\in$ N - { 1 } ).**

**<u>Solution :</u>** Let $a_n = \left(1+\frac{1}{n}\right)^n$

$$\left(1+\frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \dots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \dots + \frac{n(n-1)\dots 1}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \dots + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)}{n!}$$

=> $a_n > 2$ for n $\in$ N - { 1 }

Also, $a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} + \dots + \infty$ $\therefore$ $a_n < 3$ $\therefore$ $2 < a_n < 3$

$\therefore$ $a_n$ is a bounded sequence. - Proved.

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53. If $y = x \cdot \log\left[\frac{x}{a + bx}\right]$, then prove that $x^3 y_2 = (xy_1 - y)^2$.

**Solution :**

$y = x \cdot \log\left[\frac{x}{a + bx}\right] \Rightarrow \frac{y}{x} = \log\left[\frac{x}{a + bx}\right] \Rightarrow e^{\frac{y}{x}} = \frac{x}{a + bx} \Rightarrow e^{-\frac{y}{x}} = \frac{a}{x} + b$.

Differentiating with respect to x, $-e^{-\frac{y}{x}} \cdot \frac{xy_1 - y}{x^2} = -\frac{a}{x^2} \Rightarrow xy_1 - y = a\, e^{\frac{y}{x}}$

$\Rightarrow \log(xy_1 - y) = \log a + \frac{y}{x}$. Differentiating w.r.t. x, $\frac{xy_2 + y_1 - y_1}{xy_1 - y} = \frac{xy_1 - y}{x^2}$

$\Rightarrow \frac{xy_2}{xy_1 - y} = \frac{xy_1 - y}{x^2} \Rightarrow x^3 y_2 = (xy_1 - y)^2$. - Proved.

54. Obtain $\int \frac{x^2\,dx}{x^4 + 1}$ OR Obtain $\int \frac{2x + 3}{\sqrt{x^2 + x + 1}}\,dx$.

**Solution :** **Answer :** $\left(\frac{1}{2}\right)\left[\frac{1}{2\sqrt{2}} \tan^{-1}\frac{x^2 - 1}{\sqrt{2}\,x} + \frac{1}{2\sqrt{2}} \log\left|\frac{x^2 + 1 - \sqrt{2}\,x}{x^2 + 1 + \sqrt{2}\,x}\right|\right] + c$;

**OR** $2\sqrt{x^2 + x + 1} + 2\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + c.$

$$\int \frac{x^2\,dx}{x^4 + 1} = \left(\frac{1}{2}\right)\left[\int \frac{x^2 + 1}{x^4 + 1}\,dx + \int \frac{x^2 - 1}{x^4 + 1}\,dx\right] = \left(\frac{1}{2}\right)\left[\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx + \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx\right]$$

Let $x - \frac{1}{x} = u$ for the first integral and $x + \frac{1}{x} = v$ for the second integral

$\Rightarrow \left(1 + \frac{1}{x^2}\right)dx = du$ and $\left(1 - \frac{1}{x^2}\right)dx = dv \Rightarrow$ Integral $= \left(\frac{1}{2}\right)\left[\int \frac{du}{u^2 + 2} + \int \frac{dv}{v^2 - 2}\right]$

$$= \left(\frac{1}{2}\right)\left[\frac{1}{2\sqrt{2}} \tan^{-1}\frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log\left|\frac{v - \sqrt{2}}{v + \sqrt{2}}\right|\right] + c$$

$$= \left(\frac{1}{2}\right)\left[\frac{1}{2\sqrt{2}} \tan^{-1}\frac{x - \frac{1}{x}}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right|\right] + c$$

$$= \left(\frac{1}{2}\right)\left[\frac{1}{2\sqrt{2}} \tan^{-1}\frac{x^2 - 1}{\sqrt{2}\,x} + \frac{1}{2\sqrt{2}} \log\left|\frac{x^2 + 1 - \sqrt{2}\,x}{x^2 + 1 + \sqrt{2}\,x}\right|\right] + c.$$

$$= \left(\frac{1}{2}\right)\left[\frac{1}{2\sqrt{2}} \tan^{-1}\frac{x^2 - 1}{\sqrt{2}\,x} + \frac{1}{2\sqrt{2}} \log\left|\frac{x^2 + 1 - \sqrt{2}\,x}{x^2 + 1 + \sqrt{2}\,x}\right|\right] + c.$$

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**OR**

$$\int \frac{2x + 3}{\sqrt{x^2 + x + 1}}\,dx = \int \frac{2x + 1}{\sqrt{x^2 + x + 1}}\,dx + 2\int \frac{dx}{\sqrt{x^2 + x + 1}}$$

$$= \int \frac{\frac{d}{dx}(x^2 + x + 1)}{\sqrt{x^2 + x + 1}}\,dx + 2\int \frac{dx}{\sqrt{\left[\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]}}$$

$$= 2\sqrt{x^2 + x + 1} + 2\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + c.$$