

(1) If $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, prove that f is continuous but not differentiable at 0.

(2) Find derivatives of the following functions using the definition of a derivative.

(i) $\frac{x-1}{x+1}$ (ii) a^{3x} (iii) $\sin x^2$ (iv) $x \sin x$

[Ans: (i) $\frac{2}{(x+1)^2}$, (ii) $3a^{3x} \log a$, (iii) $2x \cos x^2$ (iv) $x \cos x + \sin x$]

(3) If $f(x) = x^2 \sin \frac{1}{x}$, $x \neq 0$ and $f(0) = 0$, prove that $f'(0) = 0$.

(4) If $f(x) = e^x - 1$, $x \geq 0$ and $f(x) = |\sin x|$, $x < 0$, is f continuous at 0? Is it differentiable at 0?

[Ans: continuous, not differentiable]

Find derivatives with respect to x of the following functions:

(5) $\frac{x^2 \sin x}{\log x}$ [Ans: $\frac{x^2 \log x \cos x + 2x \log x \sin x - x \sin x}{(\log x)^2}$]

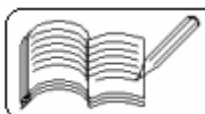
(6) $3^x e^{\log x}$ [Ans: $3^x (1 + x \log 3)$]

(7) $x^2 3^x \sin x$ [Ans: $3^x (\log 3 \cdot x^2 \sin x + 2x \sin x + x^2 \cos x)$]

(8) $\log_a (x^n)$ [Ans: $\frac{1}{x \log a}$]

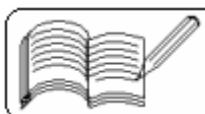
(9) $\frac{e^x}{\log x}$ [Ans: $\frac{e^x (x \log x - 1)}{x (\log x)^2}$]

(10) $\log [\log (\log x)]$ [Ans: $\frac{1}{x \log x \log (\log x)}$]



Find derivatives with respect to x of the following functions:

(11) $\log(x + \sqrt{x^2 + a^2})$	[Ans: $\frac{1}{\sqrt{x^2 + a^2}}$]
(12) $\sqrt{\frac{1-x}{1+x}}$	[Ans: $\frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}}$]
(13) $\sin[\log \cos(e^x + x^2)]$	[Ans: $-(e^x + 2x)\tan(e^x + x^2)\cos[\log \cos(e^x + x^2)]$]
(14) $\sin[\cos\{\sin(e^x + 1)\}]$	[Ans: $-e^x \cos[\cos(\sin(e^x + 1))] \cdot \sin[\sin(e^x + 1)] \cdot \cos(e^x + 1)$]
(15) $e^{\log \sin x }$	[Ans: $\cos x$ if $\sin x > 0$, $-\cos x$ if $\sin x < 0$]
(16) $e^{\tan^2 x} \cdot \sin^2 x$	[Ans: $e^{\tan^2 x}(\sin 2x + 2\tan^3 x)$]
(17) $\log \sin(\tan x^2) $	[Ans: $2x \cot(\tan x^2) \sec^2 x^2$]
(18) $\sqrt{1 - \sin 2x}$, $0 < x < \frac{\pi}{2}$	[Ans: $-\sin x - \cos x$ for $0 < x < \frac{\pi}{4}$, $\cos x + \sin x$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$, not differentiable at $x = \frac{\pi}{4}$]
(19) $\sin^{-1} \frac{x}{a}$, $0 < x < a $	[Ans: $\frac{1}{\sqrt{a^2 - x^2}}$ for $a > 0$, $\frac{-1}{\sqrt{a^2 - x^2}}$ for $a < 0$]



Find derivatives with respect to x of the following functions:

(20) $\sin^{-1} 2x\sqrt{1-x^2}$, $|x| < 1$

$$\left[\begin{array}{l} \text{Ans: } \frac{-2}{\sqrt{1-x^2}}, \text{ for } x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right) \\ \frac{2}{\sqrt{1-x^2}}, \text{ for } |x| < \frac{1}{\sqrt{2}}, \text{ not differentiable at } |x| = \frac{1}{\sqrt{2}} \end{array} \right]$$

(21) $\cos^{-1} (4x^3 - 3x)$

$$\left[\begin{array}{l} \text{Ans: } \frac{3}{\sqrt{1-x^2}} \text{ for } |x| < \frac{1}{2}, \quad \frac{-3}{\sqrt{1-x^2}} \text{ for } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \\ \text{not differentiable for } |x| = \frac{1}{2} \end{array} \right]$$

(22) $\sec^{-1} \frac{1}{2x^2 - 1}$, $0 < |x| < 1$ and $|x| \neq \frac{1}{\sqrt{2}}$

$$\left[\begin{array}{l} \text{Ans: } -\frac{2}{\sqrt{1-x^2}} \text{ for } 0 < x < 1 \text{ and } x \neq \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{1-x^2}} \text{ for } -1 < x < 0 \text{ and } x \neq -\frac{1}{\sqrt{2}} \end{array} \right]$$

(23) $\tan^{-1} \frac{\cos x}{1 - \sin x}$

$$\left[\text{Ans: } \frac{1}{2} \right]$$

(24) $\tan^{-1} \left[\frac{\sqrt{1+x^2} - 1}{x} \right]$

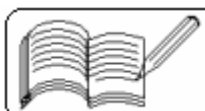
$$\left[\text{Ans: } \frac{1}{2(1+x^2)} \right]$$

(25) $\tan^{-1} \left[\frac{1 - \cos x}{1 + \cos x} \right]^{\frac{1}{2}}$ for $\pi < x < 2\pi$

$$\left[\text{Ans: } -\frac{1}{2} \right]$$

(26) $\cot^{-1} \left[\frac{\sqrt{1+x^2} - 1}{x} \right]$

$$\left[\text{Ans: } -\frac{1}{2(1+x^2)} \right]$$



Find derivatives with respect to x of the following functions:

$$(27) \sin^{-1} \frac{2x}{1+x^2}$$

$$\left[\text{Ans: } \frac{2}{1+x^2} \text{ for } |x| < 1, -\frac{2}{1+x^2} \text{ for } |x| > 1, \text{ not differentiable for } |x| = 1 \right]$$

$$(28) \sec^{-1} \frac{1+x^2}{1-x^2}$$

$$\left[\text{Ans: } \frac{2}{1+x^2} \text{ for } x \in \mathbb{R}^+ - \{1\}, -\frac{2}{1+x^2} \text{ for } x \in \mathbb{R}^+ - \{-1\}, \right. \\ \left. \text{not differentiable for } x = 0. \right]$$

$$(29) \cos^{-1} x + \cos^{-1} \sqrt{1-x^2}$$

$$\left[\text{Ans: } 0 \text{ for } x > 0, -\frac{2}{\sqrt{1-x^2}} \text{ for } x < 0, \text{ not differentiable for } x = 0. \right]$$

$$(30) \sin^{-1} \left[\frac{3 \sin x + 4 \cos x}{5} \right] \quad [\text{Ans: } \pm 1]$$

$$(31) \tan^{-1} \left[\frac{a \sin x + b \cos x}{a \cos x - b \sin x} \right] \quad [\text{Ans: } 1]$$

$$(32) \tan^{-1} \frac{2x}{1+8x^2}$$

$$\left[\text{Ans: } \frac{2(1-8x^2)}{(1+16x^2)(1+4x^2)} \right]$$

$$(33) \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}, \quad a > 0$$

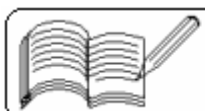
$$\left[\text{Ans: } \sqrt{a^2 - x^2} \right]$$

Solve the following problems as directed:

$$(34) \text{ Find } \frac{du}{dv}, \text{ if } u = \sin^{-1} \frac{2t}{1+t^2} \text{ and } v = \tan^{-1} \frac{2t}{1-t^2}$$

for (i) $|t| < 1$ and (ii) $t > 1$.

[Ans: (i) 1, (ii) -1]



Solve the following problems as directed:

(35) If $\frac{d}{dx}(x^n) = nx^{n-1}$ for $n \in \mathbb{N}$,

prove that $\frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{n}x^{\frac{1}{n}-1}$ ($n \in \mathbb{N}$, $x \in \mathbb{R}^+$).

(36) Find $\frac{dy}{dx}$ if $\cos(x^2 + y^2) = \log(xy)$. [Ans : - $\left(\frac{2x^2 \sin(x^2 + y^2) + 1}{2y^2 \sin(x^2 + y^2) + 1} \right) \cdot \frac{y}{x}$]

(37) If $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
[Ans : $\cot \frac{\theta}{2}$, $-\frac{1}{4a} \operatorname{cosec}^4 \frac{\theta}{2}$]

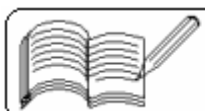
(38) If $x = \cos \theta + \cos 2\theta$ and $y = \sin \theta + \sin 2\theta$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
[Ans : $\frac{dy}{dx} = -\frac{2 \cos 2\theta + \cos \theta}{2 \sin 2\theta + \sin \theta}$, $\frac{d^2y}{dx^2} = -\frac{3(3 + 2 \cos \theta)}{(2 \sin 2\theta + \sin \theta)^3}$]

(39) If $ax^2 + 2hxy + by^2 = 0$, prove that $\frac{d^2y}{dx^2} = 0$.

(40) If $x = 3 \cos \theta - 2 \cos^3 \theta$, $y = 3 \sin \theta - 2 \sin^3 \theta$, $\theta \neq (2k - 1) \frac{\pi}{4}$, find $\frac{d^2y}{dx^2}$.
[Ans : $-\frac{1}{3} \operatorname{cosec}^3 \theta \sec 2\theta$]

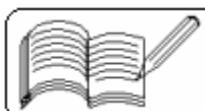
(41) Find $\frac{dy}{dx}$, if $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$. [Ans : $-\sqrt{\frac{1-y^2}{1-x^2}}$]

(42) Find $\frac{d^2y}{dx^2}$, if $x = 2 \cos t - \cos 2t$, $y = 2 \sin t - \sin 2t$,
 $t \neq 2k\pi$ or $(2k - 1) \frac{\pi}{3}$, $k \in \mathbb{Z}$. [Ans : $\frac{3}{8} \sec^3 \frac{3t}{2} \operatorname{cosec} \frac{t}{2}$]



Solve the following problems as directed:

(43) Find $\frac{dy}{dx}$, if $x = \frac{3at}{1+t^2}$, $y = \frac{3at^2}{1+t^2}$.	$\left[\text{Ans: } \frac{2t}{1-t^2} \right]$
(44) If $x = a \left(\cos \theta + \log \tan \frac{\theta}{2} \right)$, $y = a \sin \theta$, find $\frac{d^2y}{dx^2}$.	$\left[\text{Ans: } \frac{1}{a} \sec^4 \theta \sin \theta \right]$
(45) Find $\frac{d^2y}{dx^2}$, if $x = a(1 - \cos \theta)$, $y = a(\theta - \sin \theta)$, $\theta \neq k\pi$.	$\left[\text{Ans: } \frac{1}{4a} \sec^3 \frac{\theta}{2} \operatorname{cosec} \frac{\theta}{2} \right]$
(46) Find $\frac{dy}{dx}$, if $y = \sin x^x$.	$\left[\text{Ans: } x^x (1 + \log x) \cos x^x \right]$
(47) For $y = (\sin x)^x + x^{\sin x}$, find $\frac{dy}{dx}$.	$\left[\text{Ans: } (\sin x)^x (\log \sin x + x \cot x) + x^{\sin x} \left(\frac{\sin x}{x} + \cos x \log x \right) \right]$
(48) If $y = (\sqrt{x})^x + x^{\sqrt{x}}$, find $\frac{dy}{dx}$.	$\left[\text{Ans: } \frac{1}{2} (\sqrt{x})^x (1 + \log x) + \frac{1}{2} x^{\sqrt{x}} - \frac{1}{2} (\log x + 2) \right]$
(49) Find $\frac{dy}{dx}$ for $y = x^{\frac{1}{x}} + (1+x)^{\frac{1}{x}}$.	$\left[\text{Ans: } x^{\frac{1}{x}-2} (1 - \log x) + (1+x)^{\frac{1}{x}-1} \cdot \frac{x - (1+x) \log(1+x)}{x^2} \right]$
(50) Find $\frac{dy}{dx}$, if $x^m y^n = (x+y)^{m+n}$.	$\left[\text{Ans: } \frac{y}{x} \right]$



Solve the following problems as directed:

(51) $y = x^{x^x}$. Find $\frac{dy}{dx}$. [Ans : $x^{x^x} \cdot x^{x-1} (1 + x \log x + x (\log x)^2)$]

(52) If $\sin y = x \sin(a + y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.

(53) If $y = x^y$, prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$, $x \in \mathbb{R}^+$.

(54) Prove that $f(x) = |x - a|$ is not differentiable only at $x = a$. Deduce that $|x - 2| + |x - 3|$ is not differentiable only at $x = 2$ and $x = 3$.

(55) If $x = at^2$, $y = 2at$ and $t \neq 0$, then prove that $yy_2 + y_1^2 = 0$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

(56) For $x = \tan t$, $y = \tan pt$, prove that $(1 + x^2)y_2 + 2xy_1 = 2pyy_1$.

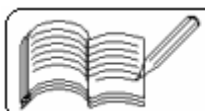
(57) If $y = e^{m \sin^{-1} x}$, prove that $(1 - x^2)y_2 - xy_1 = m^2 y$, where $m \neq 0$.

(58) If $y = e^x (\cos x + \sin x)$, prove that $y_2 - 2y_1 + 2y = 0$.

(59) If $y = (x + \sqrt{x^2 + 1})^m$, prove that $(1 + x^2)y_2 + xy_1 = m^2 y$.

(60) If $y = (\cos^{-1} x)^2$, prove that $(1 - x^2)y_2 - xy_1 = 2$.

(61) If $y = \sin(m \sin^{-1} x)$, prove that $(1 - x^2)y_2 - xy_1 + m^2 y = 0$.



Solve the following problems as directed:

$$(62) \text{ If } y = \sin pt, \quad x = \sin t, \quad \text{prove that } (1 - x^2)y_2 - xy_1 + p^2y = 0.$$

$$(63) \text{ If } y = e^{m \tan^{-1} x}, \quad \text{prove that } (1 + x^2)y_2 + (2x - m)y_1 = 0.$$

$$(64) \text{ If } 2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}, \quad \text{prove that } (x^2 - 1)y_2 + xy_1 = m^2y.$$

$$(65) \text{ If } y = e^{\sqrt{x}} + e^{-\sqrt{x}}, \quad \text{prove that } 4xy_2 + 2y_1 - y = 0.$$

$$(66) \text{ If } \cos^{-1} \frac{y}{b} = \log \left(\frac{x}{n} \right)^n, \quad \text{prove that } x^2y_2 + xy_1 + n^2y = 0.$$

Find derivatives with respect to x of the following functions:

$$(67) \quad x \sin^{-1} \frac{2x}{1+x^2}, \quad |x| < 1$$

$$\left[\text{Ans: } \frac{2x}{1+x^2} + 2 \tan^{-1} x \right]$$

$$(68) \quad \frac{x \cos^{-1} x}{\sqrt{1-x^2}}$$

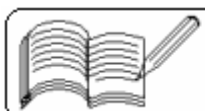
$$\left[\text{Ans: } \frac{\cos^{-1} x}{(1-x^2)^{\frac{3}{2}}} - \frac{x}{1-x^2} \right]$$

$$(69) \quad \tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$$

$$[\text{Ans: } -1]$$

$$(70) \quad \frac{(x^3 - 2)\sqrt{x^2 + 1}}{(x^2 + 2x + 3)(2x - 5)^{\frac{3}{2}}}$$

$$\left[\text{Ans: } \frac{(x^3 - 2)\sqrt{x^2 + 1}}{(x^2 + 2x + 3)(2x - 5)^{\frac{3}{2}}} \left[\frac{3x^2}{x^3 - 2} + \frac{x}{x^2 + 1} - \frac{2(x + 1)}{x^2 + 2x + 3} - \frac{3}{2x - 5} \right] \right]$$



Find derivatives with respect to x of the following functions:

(71) $(\sin x)^{\log x} + (\log x)^x$

$$\left[\text{Ans: } (\sin x)^{\log x} \left(\log x \cot x + \frac{\log(\sin x)}{x} \right) + (\log x)^x \left(\frac{1}{\log x} + \log \log x \right) \right]$$

(72) $\frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right]$

$$\left[\text{Ans: } \frac{1}{a + b \cos x} \right]$$

Solve the following problems as directed:

(73) Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for all x . If $f'(0)$ exists, find its value.

[Ans: 0]

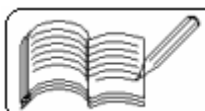
(74) If $x = \sec \theta - \cos \theta$ and $y = \sec^n \theta - \cos^n \theta$,
then show that $(x^2 + 4) \left(\frac{dy}{dx} \right)^2 = (y^2 + 4)n^2$.

(75) If $x = \cos \theta$, $y = \sin^3 \theta$, prove that $\left(\frac{dy}{dx} \right)^2 + y \left(\frac{d^2y}{dx^2} \right) = 3 \sin^2 \theta (5 \cos^2 \theta - 1)$.

(76) If $y^2 = p(x)$, then prove that $2 \frac{d}{dx} \left(y^3 \frac{d^2y}{dx^2} \right) = p(x)p'''(x)$.

(77) If u and v are derivable functions of x ,
then prove that $\frac{d}{dx}(u^v) = v u^{v-1} \frac{du}{dx} + u^v \frac{dv}{dx} \log u$.

(78) If $f(2) = 4$, $g(2) = 9$, $f'(2) = g'(2)$, then find $\lim_{x \rightarrow 2} \frac{\sqrt{f(x)} - 2}{\sqrt{g(x)} - 2}$. [Ans: $\frac{3}{2}$]



Solve the following problems as directed:

(79) Prove that $\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^3$.

(80) If $\tan \frac{y}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{x}{2}$, prove that $\frac{dy}{dx} = \frac{\sin y}{\sin x} = \frac{\sqrt{1-e^2}}{1+e \cos x}$.

(81) If $ky = \sin(x+y)$, prove that $y_2 = -y(1+y_1)^3$.

(82) If $\log y = \log \sin x - x^2$, prove that $y_2 + 4xy_1 + (4x^2 + 3)y = 0$.

(83) For $y = \log_7(\log_7 x^4)$, obtain $\frac{dy}{dx}$. [Ans: $\frac{1}{x(\log 7)(\log x)}$]

(84) If $\frac{x}{x-y} = \log \frac{a}{x-y}$, prove that $\frac{dy}{dx} = 2 - \frac{x}{y}$.

(85) Prove that $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, ($a \neq 0$) $\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

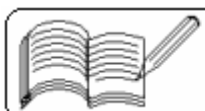
(86) For $y = \tan^{-1} \frac{3xa^2 - x^3}{a(a^2 - 3x^2)}$, obtain $\frac{dy}{dx}$ [Ans: $\frac{3a}{a^2 + x^2}$]

(87) Prove that $y = x \log [(ax)^{-1} + a^{-1}] \Rightarrow x(x+1)y_2 + xy_1 = y - 1$.

(88) If $y = x \log \left(\frac{x}{a+bx} \right)$, prove that $x^3 y_2 = (y - xy_1)^2$.

(89) If $y = \sqrt{x+1} + \sqrt{x-1}$, prove that $(x^2 - 1)y_2 + xy_1 = \frac{y}{4}$.

(90) Differentiate $\sin^{-1} x$ w.r.t. x , $|x| < 1$ using the definition of derivative.



Solve the following problems as directed:

(91) If $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$,
prove that $(x^2 - 1)y_2 + xy_1 - n^2y = 0$.

(92) If $g(x_1 + x_2) = g(x_1)g(x_2)$ and $g(x) \neq 0 \forall x \in D_g$ and $g'(0) = 2$,
then prove that $g'(x) = 2g(x)$.

(93) If $f^{-1} = g$ and $f'(x) = \frac{1}{1+x^3}$, then prove that $g'(y) = 1 + [g(y)]^3$.

(94) If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then prove that
$$\lim_{x \rightarrow a} \frac{g(x)f(a) - f(x)g(a)}{x - a} = 5.$$

(95) For $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, prove that $p^4 + p^3 \frac{d^2p}{d\theta^2} = a^2 b^2$.

(96) If $(a - b \cos y)(a + b \cos x) = a^2 - b^2$, prove that $\frac{dy}{dx} = \frac{(a^2 - b^2)^{\frac{1}{2}}}{a + b \cos x}$.

(97) If $S_n = a + ax + ax^2 + \dots$ upto n terms,
show that $(1 - x) \frac{d}{dx} S_n = n S_{n-1} - (n - 1) S_n$.

(98) P.t. $y = x \sin y \Rightarrow \frac{dy}{dx} = \frac{\sin y}{1 - x \cos y} = \frac{y}{x(1 - x \cos y)} = \frac{\sin^2 y}{\sin y - y \cos y}$.

(99) If $y = f(x)$ is one-one and onto, p.t. $f''(x) = -(f^{-1})'' [f'(x)]^3$.

(100) Find $\frac{dy}{dx}$ for $x = e^{\tan^{-1} \left[\frac{y - x^2}{x^2} \right]}$. [Ans: $x \left(\frac{y^2}{x^4} + 2 \right)]$

