(1) If $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, prove that f is continuous but not differentiable at 0.

(2) Find derivatives of the following functions using the definition of a derivative.

(i) $\frac{x - 1}{x + 1}$ (ii) a^{3x} (iii) $\sin x^2$ (iv) $x \sin x$

[Ans: (i) $\frac{2}{(x + 1)^2}$, (ii) $3 a^{3x} \log a$, (iii) $2x \cos x^2$ (iv) $x \cos x + \sin x$]

(3) If $f(x) = x^2 \sin \frac{1}{x}$, $x \neq 0$ and $f(0) = 0$, prove that $f'(0) = 0$.

(4) If $f(x) = e^x - 1$, $x \geq 0$ and $f(x) = |\sin x|$, $x < 0$, is f continuous at 0 ? Is it differentiable at 0 ?

[Ans: continuous, not differentiable]

Find derivatives with respect to x of the following functions:

(5) $\frac{x^2 \sin x}{\log x}$ [Ans : $\frac{x^2 \log x \cos x + 2x \log x \sin x - x \sin x}{(\log x)^2}$]

(6) $3^x e^{\log x}$ [Ans : $3^x (1 + x \log 3)$]

(7) $x^2 3^x \sin x$ [Ans: $3^x (\log 3 . x^2 \sin x + 2x \sin x + x^2 \cos x)$]

(8) $\log_{a^n} (x^n)$ [Ans : $\frac{1}{x \log a}$]

(9) $\frac{e^x}{\log x}$ [Ans : $\frac{e^x (x \log x - 1)}{x (\log x)^2}$]

(10) $\log [\log (\log x)]$ [Ans : $\frac{1}{x \log x \log (\log x)}$]

Find derivatives with respect to x of the following functions:

(11) $\log\left(x + \sqrt{x^2 + a^2}\right)$

$\left[\text{Ans: } \dfrac{1}{\sqrt{x^2 + a^2}}\right]$

(12) $\sqrt{\dfrac{1 - x}{1 + x}}$

$\left[\text{Ans: } \dfrac{-1}{(1 - x)^{\frac{1}{2}} (1 + x)^{\frac{3}{2}}}\right]$

(13) $\sin[\log|\cos(e^x + x^2)|]$

$[\text{Ans: } -(e^x + 2x)\tan(e^x + x^2)\cos[\log|\cos(e^x + x^2)|]]$

(14) $\sin[\cos\{\sin(e^x + 1)|]$

$\left[\text{Ans: } -e^x \cos\left[\cos\left(\sin(e^x + 1)\right)\right] . \sin\left[\sin(e^x + 1)\right] . \cos(e^x + 1)\right]$

(15) $e^{\log|\sin x|}$

$[\text{Ans: } \cos x \text{ if } \sin x > 0, \ -\cos x \text{ if } \sin x < 0]$

(16) $e^{\tan^2 x} \cdot \sin^2 x$

$\left[\text{Ans: } e^{\tan^2 x}(\sin 2x + 2\tan^3 x)\right]$

(17) $\log|\sin(\tan x^2)|$

$[\text{Ans: } 2x\cot(\tan x^2)\sec^2 x^2]$

(18) $\sqrt{1 - \sin 2x}, \quad 0 < x < \dfrac{\pi}{2}$

$\left[\text{Ans: } -\sin x - \cos x \text{ for } 0 < x < \dfrac{\pi}{4}, \quad \cos x + \sin x \text{ for } \dfrac{\pi}{4} < x < \dfrac{\pi}{2}, \right.$

$\left. \text{not differentiable at } x = \dfrac{\pi}{4}\right]$

(19) $\sin^{-1}\dfrac{x}{a}, \quad 0 < |x| < |a|$

$\left[\text{Ans: } \dfrac{1}{\sqrt{a^2 - x^2}} \text{ for } a > 0, \quad \dfrac{-1}{\sqrt{a^2 - x^2}} \text{ for } a < 0\right]$

Find derivatives with respect to x of the following functions:

(20) $\sin^{-1} 2x\sqrt{1 - x^2}\ , \quad |x| < 1$

$$\left[\text{Ans: } \frac{-2}{\sqrt{1-x^2}}, \text{ for } x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right) \quad \frac{2}{\sqrt{1-x^2}}, \text{ for } |x| < \frac{1}{\sqrt{2}}, \text{ not differentiable at } |x| = \frac{1}{\sqrt{2}} \right]$$

(21) $\cos^{-1}(4x^3 - 3x)$

$$\left[\text{Ans: } \frac{3}{\sqrt{1-x^2}} \text{ for } |x| < \frac{1}{2}, \quad \frac{-3}{\sqrt{1-x^2}} \text{ for } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \quad \text{not differentiable for } |x| = \frac{1}{2} \right]$$

(22) $\sec^{-1} \dfrac{1}{2x^2 - 1}\ , \quad 0 < |x| < 1 \text{ and } |x| \neq \dfrac{1}{\sqrt{2}}$

$$\left[\text{Ans: } -\frac{2}{\sqrt{1-x^2}} \text{ for } 0 < x < 1 \text{ and } x \neq \frac{1}{\sqrt{2}} \quad \frac{2}{\sqrt{1-x^2}} \text{ for } -1 < x < 0 \text{ and } x \neq -\frac{1}{\sqrt{2}} \right]$$

(23) $\tan^{-1} \dfrac{\cos x}{1 - \sin x}$

$$\left[\text{Ans: } \frac{1}{2} \right]$$

(24) $\tan^{-1} \left[\dfrac{\sqrt{1 + x^2} - 1}{x} \right]$

$$\left[\text{Ans: } \frac{1}{2(1 + x^2)} \right]$$

(25) $\tan^{-1} \left[\dfrac{1 - \cos x}{1 + \cos x} \right]^{\frac{1}{2}} \quad \text{for } \pi < x < 2\pi$

$$\left[\text{Ans: } -\frac{1}{2} \right]$$

(26) $\cot^{-1} \left[\dfrac{\sqrt{1 + x^2} - 1}{x} \right]$

$$\left[\text{Ans: } -\frac{1}{2(1 + x^2)} \right]$$

Find derivatives with respect to x of the following functions:

(27) $\sin^{-1}\dfrac{2x}{1+x^2}$

$\left[\text{Ans: } \dfrac{2}{1+x^2} \text{ for } |x| < 1,\ -\dfrac{2}{1+x^2} \text{ for } |x| > 1, \text{ not differentiable for } |x| = 1\right]$

(28) $\sec^{-1}\dfrac{1+x^2}{1-x^2}$

$\left[\text{Ans: } \dfrac{2}{1+x^2} \text{ for } x \in R^{+} - \{1\},\ -\dfrac{2}{1+x^2} \text{ for } x \in R^{+} - \{-1\}, \text{ not differentiable for } x = 0.\right]$

(29) $\cos^{-1}x + \cos^{-1}\sqrt{1-x^2}$

$\left[\text{Ans: } 0 \text{ for } x > 0,\ -\dfrac{2}{\sqrt{1-x^2}} \text{ for } x < 0, \text{ not differentiable for } x = 0.\right]$

(30) $\sin^{-1}\left[\dfrac{3\sin x + 4\cos x}{5}\right]$ $[\text{Ans: } \pm 1]$

(31) $\tan^{-1}\left[\dfrac{a\sin x + b\cos x}{a\cos x - b\sin x}\right]$ $[\text{Ans: } 1]$

(32) $\tan^{-1}\dfrac{2x}{1+8x^2}$ $\left[\text{Ans: } \dfrac{2(1-8x^2)}{(1+16x^2)(1+4x^2)}\right]$

(33) $\dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$, $a > 0$ $\left[\text{Ans: } \sqrt{a^2-x^2}\right]$

Solve the following problems as directed:

(34) Find $\dfrac{du}{dv}$, if $u = \sin^{-1}\dfrac{2t}{1+t^2}$ and $v = \tan^{-1}\dfrac{2t}{1-t^2}$

for (i) $|t| < 1$ and (ii) $t > 1$. [Ans: (i) 1, (ii) -1]

Solve the following problems as directed:

(35) If $\frac{d}{dx}(x^n) = n x^{n-1}$ for $n \in N$,

prove that $\frac{d}{dx}\left(x^{\frac{1}{n}}\right) = \frac{1}{n} x^{\frac{1}{n}-1}$ $(n \in N,\ \ x \in R^+)$.

(36) Find $\frac{dy}{dx}$ if $\cos(x^2 + y^2) = \log(xy)$. $\left[\text{Ans : } -\left(\frac{2x^2 \sin(x^2 + y^2) + 1}{2y^2 \sin(x^2 + y^2) + 1}\right) \cdot \frac{y}{x}\right]$

(37) If $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\left[\text{Ans : } \cot\frac{\theta}{2},\ \ -\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}\right]$

(38) If $x = \cos\theta + \cos 2\theta$ and $y = \sin\theta + \sin 2\theta$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\left[\text{Ans : } \frac{dy}{dx} = -\frac{2\cos 2\theta + \cos\theta}{2\sin 2\theta + \sin\theta},\ \ \frac{d^2y}{dx^2} = -\frac{3(3 + 2\cos\theta)}{(2\sin 2\theta + \sin\theta)^3}\right]$

(39) If $ax^2 + 2hxy + by^2 = 0$, prove that $\frac{d^2y}{dx^2} = 0$.

(40) If $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$, $\theta \neq (2k - 1)\frac{\pi}{4}$, find $\frac{d^2y}{dx^2}$.

$\left[\text{Ans : } -\frac{1}{3}\operatorname{cosec}^3\theta \sec 2\theta\right]$

(41) Find $\frac{dy}{dx}$, if $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$. $\left[\text{Ans : } -\sqrt{\frac{1 - y^2}{1 - x^2}}\right]$

(42) Find $\frac{d^2y}{dx^2}$, if $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$,

$t \neq 2k\pi$ or $(2k - 1)\frac{\pi}{3}$, $k \in Z$. $\left[\text{Ans : } \frac{3}{8}\sec^3\frac{3t}{2}\operatorname{cosec}\frac{t}{2}\right]$

Solve the following problems as directed:

(43) Find $\frac{dy}{dx}$, if $x = \frac{3at}{1 + t^2}$, $y = \frac{3at^2}{1 + t^2}$. $\left[\text{Ans : } \frac{2t}{1 - t^2}\right]$

(44) If $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$, find $\frac{d^2y}{dx^2}$. $\left[\text{Ans : } \frac{1}{a}\sec^4\theta\sin\theta\right]$

(45) Find $\frac{d^2y}{dx^2}$, if $x = a(1 - \cos\theta)$, $y = a(\theta - \sin\theta)$, $\theta \neq k\pi$.
$\left[\text{Ans : } \frac{1}{4a}\sec^3\frac{\theta}{2}\operatorname{cosec}\frac{\theta}{2}\right]$

(46) Find $\frac{dy}{dx}$, if $y = \sin x^x$. $\left[\text{Ans : } x^x(1 + \log x)\cos x^x\right]$

(47) For $y = (\sin x)^x + x^{\sin x}$, find $\frac{dy}{dx}$.
$\left[\text{Ans : } (\sin x)^x(\log\sin x + x\cot x) + x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right)\right]$

(48) If $y = (\sqrt{x})^x + x^{\sqrt{x}}$, find $\frac{dy}{dx}$.
$\left[\text{Ans : } \frac{1}{2}(\sqrt{x})^x(1 + \log x) + \frac{1}{2}x^{\sqrt{x} - \frac{1}{2}}(\log x + 2)\right]$

(49) Find $\frac{dy}{dx}$ for $y = x^{\frac{1}{x}} + (1 + x)^{\frac{1}{x}}$.
$\left[\text{Ans : } x^{\frac{1}{x} - 2}(1 - \log x) + (1 + x)^{\frac{1}{x} - 1}\cdot\frac{x - (1 + x)\log(1 + x)}{x^2}\right]$

(50) Find $\frac{dy}{dx}$, if $x^m y^n = (x + y)^{m + n}$. $\left[\text{Ans : } \frac{y}{x}\right]$

Solve the following problems as directed:

(51) $y = x^{x^x}$. Find $\frac{dy}{dx}$. $\left[\text{Ans : } x^{x^x} \cdot x^{x-1}(1 + x\log x + x(\log x)^2) \right]$

(52) If $\sin y = x\sin(a + y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.

(53) If $y = x^y$, prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$, $x \in R^+$.

(54) Prove that $f(x) = |x - a|$ is not differentiable only at $x = a$. Deduce that $|x - 2| + |x - 3|$ is not differentiable only at $x = 2$ and $x = 3$.

(55) If $x = at^2$, $y = 2at$ and $t \neq 0$, then prove that $y y_2 + y_1^2 = 0$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

(56) For $x = \tan t$, $y = \tan pt$, prove that $(1 + x^2)y_2 + 2xy_1 = 2pyy_1$.

(57) If $y = e^{m\sin^{-1}x}$, prove that $(1 - x^2)y_2 - xy_1 = m^2 y$, where $m \neq 0$.

(58) If $y = e^x(\cos x + \sin x)$, prove that $y_2 - 2y_1 + 2y = 0$.

(59) If $y = (x + \sqrt{x^2 + 1})^m$, prove that $(1 + x^2)y_2 + xy_1 = m^2y$.

(60) If $y = (\cos^{-1}x)^2$, prove that $(1 - x^2)y_2 - xy_1 = 2$.

(61) If $y = \sin(m\sin^{-1}x)$, prove that $(1 - x^2)y_2 - xy_1 + m^2y = 0$.

Solve the following problems as directed:

(62) If $y = \sin pt$, $x = \sin t$, prove that $(1 - x^2)y_2 - xy_1 + p^2 y = 0$.

(63) If $y = e^{m\tan^{-1}x}$, prove that $(1 + x^2)y_2 + (2x - m)y_1 = 0$.

(64) If $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$, prove that $(x^2 - 1)y_2 + xy_1 = m^2 y$.

(65) If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$, prove that $4xy_2 + 2y_1 - y = 0$.

(66) If $\cos^{-1}\frac{y}{b} = \log\left(\frac{x}{n}\right)^n$, prove that $x^2 y_2 + xy_1 + n^2 y = 0$.

Find derivatives with respect to x of the following functions:

(67) $x\sin^{-1}\frac{2x}{1+x^2}$, $|x| < 1$ $\left[\text{Ans: } \frac{2x}{1+x^2} + 2\tan^{-1}x\right]$

(68) $\frac{x\cos^{-1}x}{\sqrt{1 - x^2}}$ $\left[\text{Ans: } \frac{\cos^{-1}x}{(1 - x^2)^{\frac{3}{2}}} - \frac{x}{1 - x^2}\right]$

(69) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ [Ans: -1]

(70) $\frac{(x^3 - 2)\sqrt{x^2 + 1}}{(x^2 + 2x + 3)(2x - 5)^{\frac{3}{2}}}$

$$\left[\text{Ans: } \frac{(x^3 - 2)\sqrt{x^2 + 1}}{(x^2 + 2x + 3)(2x - 5)^{\frac{3}{2}}}\left[\frac{3x^2}{x^3 - 2} + \frac{x}{x^2 + 1} - \frac{2(x + 1)}{x^2 + 2x + 3} - \frac{3}{2x - 5}\right]\right]$$

Find derivatives with respect to x of the following functions:

(71) $(\sin x)^{\log x} + (\log x)^{x}$

$$\left[\text{Ans: } (\sin x)^{\log x}\left(\log x \cot x + \frac{\log(\sin x)}{x} \right) + (\log x)^{x}\left(\frac{1}{\log x} + \log\log x \right) \right]$$

(72) $\dfrac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\sqrt{\dfrac{a - b}{a + b}} \tan\dfrac{x}{2} \right]$ $\left[\text{Ans: } \dfrac{1}{a + b\cos x} \right]$

Solve the following problems as directed:

(73) Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for all x. If $f'(0)$ exists, find its value.

[Ans: 0]

(74) If $x = \sec\theta - \cos\theta$ and $y = \sec^{n}\theta - \cos^{n}\theta$, then show that $(x^2 + 4)\left(\dfrac{dy}{dx}\right)^2 = (y^2 + 4)n^2$.

(75) If $x = \cos\theta$, $y = \sin^3\theta$, prove that $\left(\dfrac{dy}{dx}\right)^2 + y\left(\dfrac{d^2y}{dx^2}\right) = 3\sin^2\theta\,(5\cos^2\theta - 1)$.

(76) If $y^2 = p(x)$, then prove that $2\dfrac{d}{dx}\left(y^3 \dfrac{d^2y}{dx^2} \right) = p(x)p'''(x)$.

(77) If u and v are derivable functions of x, then prove that $\dfrac{d}{dx}(u^v) = v u^{v-1}\dfrac{du}{dx} + u^v \dfrac{dv}{dx}\log u$.

(78) If $f(2) = 4$, $g(2) = 9$, $f'(2) = g'(2)$, then find $\lim\limits_{x \to 2} \dfrac{\sqrt{f(x)} - 2}{\sqrt{g(x)} - 2}$. $\left[\text{Ans: } \dfrac{3}{2} \right]$

Solve the following problems as directed:

(79) Prove that $\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^3$.

(80) If $\tan\frac{y}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{x}{2}$, prove that $\frac{dy}{dx} = \frac{\sin y}{\sin x} = \frac{\sqrt{1-e^2}}{1+e\cos x}$.

(81) If $ky = \sin(x+y)$, prove that $y_2 = -y(1+y_1)^3$.

(82) If $\log y = \log\sin x - x^2$, prove that $y_2 + 4xy_1 + (4x^2+3)y = 0$.

(83) For $y = \log_7(\log_7 x^4)$, obtain $\frac{dy}{dx}$. $\left[\text{Ans: } \frac{1}{x(\log 7)(\log x)}\right]$

(84) If $\frac{x}{x-y} = \log\frac{a}{x-y}$, prove that $\frac{dy}{dx} = 2 - \frac{x}{y}$.

(85) Prove that $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, $(a \neq 0) \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

(86) For $y = \tan^{-1}\frac{3xa^2 - x^3}{a(a^2-3x^2)}$, obtain $\frac{dy}{dx}$ $\left[\text{Ans: } \frac{3a}{a^2+x^2}\right]$

(87) Prove that $y = x\log[(ax)^{-1} + a^{-1}] \Rightarrow x(x+1)y_2 + xy_1 = y - 1$.

(88) If $y = x\log\left(\frac{x}{a+bx}\right)$, prove that $x^3y_2 = (y - xy_1)^2$.

(89) If $y = \sqrt{x+1} + \sqrt{x-1}$, prove that $(x^2-1)y_2 + xy_1 = \frac{y}{4}$.

(90) Differentiate $\sin^{-1}x$ w.r.t. x, $|x| < 1$ using the definition of derivative.

Solve the following problems as directed:

(91) If $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$,
prove that $(x^2 - 1)y_2 + xy_1 - n^2 y = 0$.

(92) If $g(x_1 + x_2) = g(x_1)\,g(x_2)$ and $g(x) \neq 0\ \forall\ x \in D_g$ and $g'(0) = 2$, then prove that $g'(x) = 2g(x)$.

(93) If $f^{-1} = g$ and $f'(x) = \dfrac{1}{1 + x^3}$, then prove that $g'(y) = 1 + [g(y)]^3$.

(94) If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then prove that
$\lim\limits_{x \to a} \dfrac{g(x)f(a) - f(x)g(a)}{x - a} = 5$.

(95) For $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, prove that $p^4 + p^3 \dfrac{d^2p}{d\theta^2} = a^2 b^2$.

(96) If $(a - b\cos y)(a + b\cos x) = a^2 - b^2$, prove that $\dfrac{dy}{dx} = \dfrac{(a^2 - b^2)^{\frac{1}{2}}}{a + b\cos x}$.

(97) If $S_n = a + ax + ax^2 + \dots$ upto n terms,
show that $(1 - x)\dfrac{d}{dx} S_n = nS_{n-1} - (n - 1)S_n$.

(98) P.t. $y = x\sin y \Rightarrow \dfrac{dy}{dx} = \dfrac{\sin y}{1 - x\cos y} = \dfrac{y}{x(1 - x\cos y)} = \dfrac{\sin^2 y}{\sin y - y\cos y}$.

(99) If $y = f(x)$ is one-one and onto, p.t. $f''(x) = -(f^{-1})''[f'(x)]^3$.

(100) Find $\dfrac{dy}{dx}$ for $x = e^{\tan^{-1}\left[\frac{y - x^2}{x^2}\right]}$. $\left[\text{Ans}: x\left(\dfrac{y^2}{x^4} + 2\right)\right]$